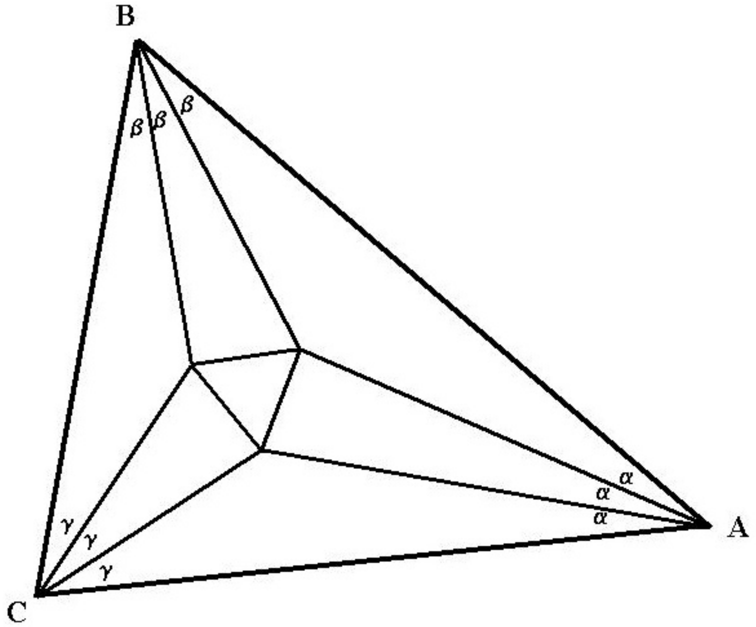
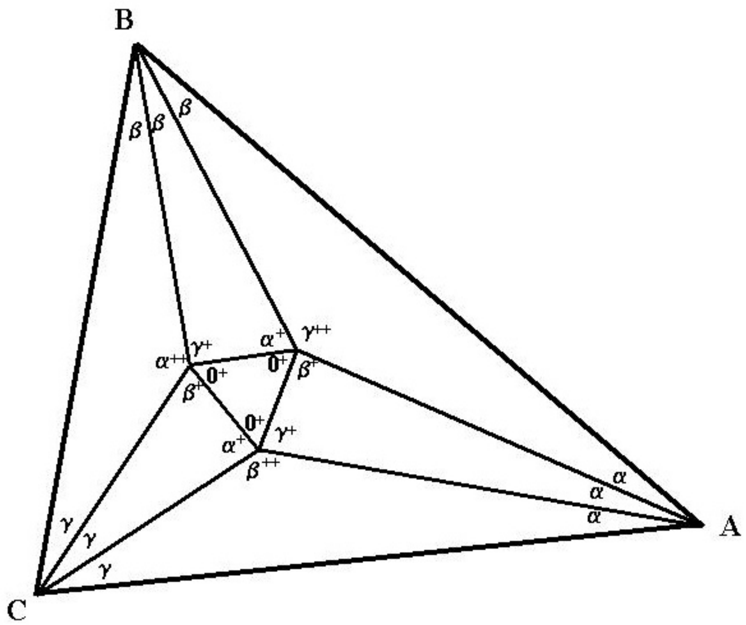
Proof of Morley's Theorem

Morley's Theorem states that if you trisect the angles of any triangle then the lines meet at the vertices of an equilateral triangle.

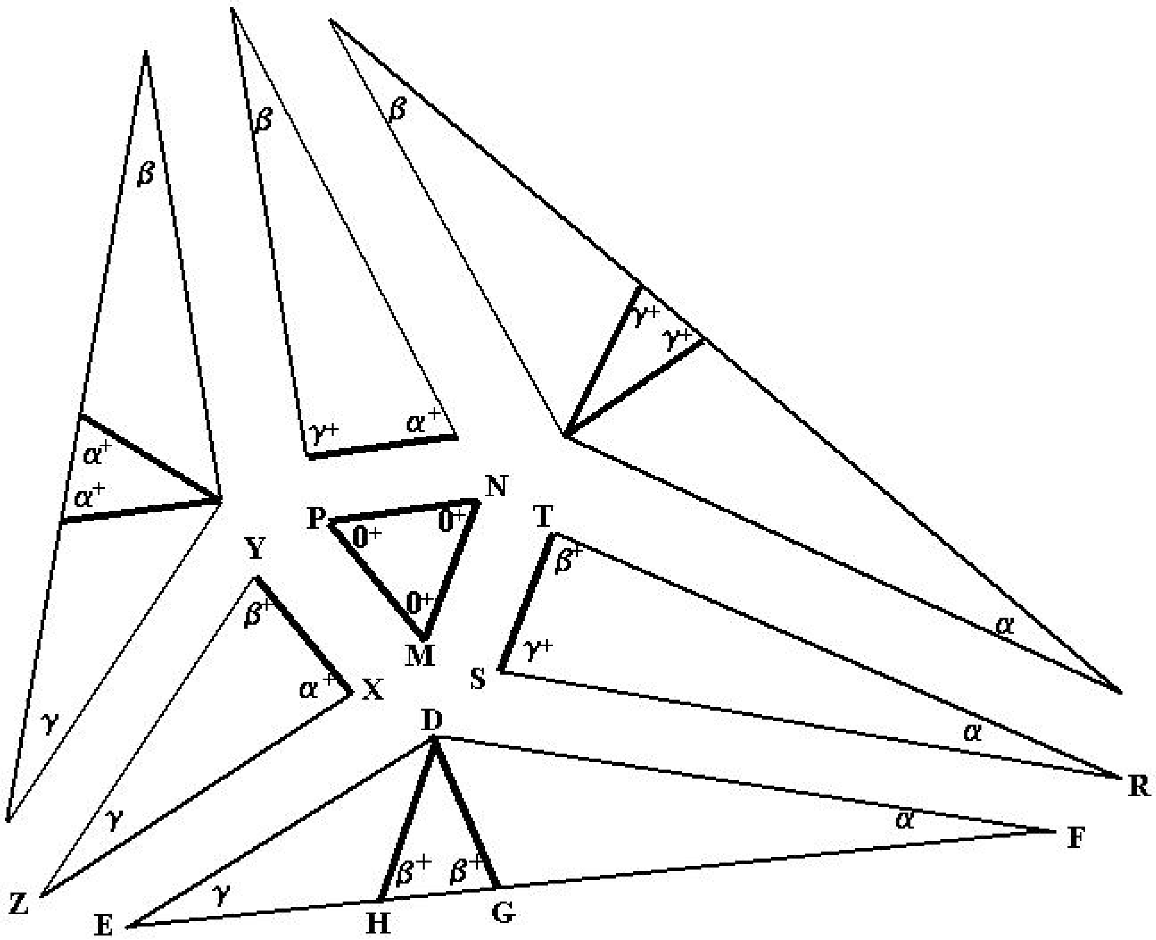


Note that $\alpha + \beta + \gamma = 60$. The theorem dates from around 1899, a direct proof is hard, the proof below is an easy one based on the work of Conway and someone else (Doyle?) dating from around 1995.

For any angle θ let $\theta^+ = \theta + 60$. Then we would like to prove that all the angles are as follows (in particular the angles in the middle triangle are all $0^+ = 60$ degrees):



Instead of working forward, work backwards. Show that from an equilateral triangle you can construct a triangle with any angles, i.e. with arbitrary α, β and γ (with sum 60, of course). Start with an arbitrary equilateral triangle. Construct six more triangles as shown with angles as indicated and the twelve emboldened sides of equal length, i.e. $MP = XY = DG = DH = MN = ST$, etc. This can be done because the triangles DGH, etc., are isosceles.



Triangles DGE and XYZ are similar and $DG = XY$ so the two triangles are in fact equivalent, and in particular $DE = XZ$. Similarly $DF = SR$, etc. Note that angle $EDF = 180 - \alpha - \gamma = \beta^{++}$. Then since the angles $PMN + YXZ + EDF + TSR = 0^{+} + \alpha^{+} + \beta^{++} + \gamma^{+} = 360$ and since the sides match we see that triangles MNP, XYZ, DEF and RST fit perfectly together, and similarly for the other triangles. Hence the conclusion.